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SUPERCONGRUENCES INVOLVING LUCAS SEQUENCES

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ABSTRACT. For $A, B \in \mathbb{Z}$, the Lucas sequence $u_n(A, B)$ ($n = 0, 1, 2, \dots$) are defined by $u_0(A, B) = 0$, $u_1(A, B) = 1$, and $u_{n+1}(A, B) = Au_n(A, B) - Bu_{n-1}(A, B)$ ($n = 1, 2, 3, \dots$). For any odd prime p and positive integer n , we establish the new result

$$\frac{u_{pn}(A, B) - \left(\frac{A^2-4B}{p}\right)u_n(A, B)}{pn} \in \mathbb{Z}_p,$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol and \mathbb{Z}_p is the ring of p -adic integers.

Let p be an odd prime and let n be a positive integer. For any integer $m \not\equiv 0 \pmod{p}$, we show that

$$\frac{1}{pn} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{m^k} - \left(\frac{\Delta}{p}\right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{m^r} \right) \in \mathbb{Z}_p$$

and furthermore

$$\frac{1}{n} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{m^k} - \left(\frac{\Delta}{p}\right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{m^r} \right) \equiv \frac{\binom{2n-1}{n-1}}{m^{n-1}} u_{p-\left(\frac{\Delta}{p}\right)}(m-2, 1) \pmod{p^2},$$

where $\Delta = m(m-4)$. In particular, we have

$$\frac{1}{n} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{2^k} - \left(\frac{-1}{p}\right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{2^r} \right) \equiv 0 \pmod{p^2},$$

and

$$\frac{1}{n} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{3^k} - \left(\frac{p}{3}\right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{3^r} \right) \equiv 0 \pmod{p^2} \text{ if } p > 3.$$

We also pose some conjectures for further research.

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1. INTRODUCTION

Let $p > 3$ be a prime. In 2006 H. Pan and the author [PS] deduced from a sophisticated combinatorial identity the congruence

$$\sum_{k=0}^{p-1} \binom{2k}{k+d} \equiv \left(\frac{p-d}{3} \right) \pmod{p} \quad \text{for } d = 0, \dots, p-1,$$

where $(-)$ denotes the Legendre symbol. In 2011 the author and R. Tauraso [ST11] showed further that

$$\sum_{k=0}^{p-1} \binom{2k}{k+d} \equiv \left(\frac{p-d}{3} \right) \pmod{p^2} \quad \text{for } d = 0, 1. \quad (1.1)$$

Recently, J.-C. Liu [L16] proved the following extension with $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ conjectured by M. Apagodu and D. Zeilberger [AZ]:

$$\sum_{k=0}^{pn-1} \binom{2k}{k} \equiv \left(\frac{p}{3} \right) \sum_{k=0}^{n-1} \binom{2k}{k} \pmod{p^2} \quad (1.2)$$

and

$$\sum_{k=0}^{pn-1} C_k \equiv \begin{cases} \sum_{r=0}^{n-1} C_r \pmod{p^2} & \text{if } p \equiv 1 \pmod{3}, \\ -\sum_{r=0}^{n-1} (3r+2)C_r \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \quad (1.3)$$

where C_k denotes the Catalan number $\binom{2k}{k} - \binom{2k}{k+1} = \binom{2k}{k}/(k+1)$. Note that this result in the case $n = 1$ yields the supercongruence (1.1).

For given integers A and B , the Lucas sequence $u_n = u_n(A, B)$ ($n \in \mathbb{N} = \{0, 1, 2, \dots\}$) is given by

$$u_0 = 0, \quad u_1 = 1, \quad \text{and } u_{n+1} = Au_n - Bu_{n-1} \text{ for } n = 1, 2, 3, \dots$$

It is well known that $p \mid u_{p-(\frac{A^2-4B}{p})}$ for any odd prime p not dividing B (see, e.g., [S10, Lemma 2.3]). In 2010 the author [S10] showed that for any nonzero integer m and odd prime p not dividing m we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m(m-4)}{p} \right) + u_{p-(\frac{m(m-4)}{p})}(m-2, 1) \pmod{p^2}. \quad (1.4)$$

In this paper we obtain the following general result which is a common extension of (1.1)-(1.4).

Theorem 1.1. *Let $m \in \mathbb{Z} \setminus \{0\}$, $n \in \mathbb{Z}^+$ and $\Delta = m(m-4)$. For any odd prime p not dividing m , we have*

$$\frac{1}{n} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{m^k} - \left(\frac{\Delta}{p} \right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{m^r} \right) \equiv \frac{\binom{2n-1}{n-1}}{m^{n-1}} u_{p-(\frac{\Delta}{p})}(m-2, 1) \pmod{p^2} \quad (1.5)$$

and

$$\begin{aligned} & \frac{1}{n} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k+1}}{m^k} - \left(\frac{\Delta}{p} \right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r+1}}{m^r} + \left(\frac{m}{2} - \frac{\binom{2n-1}{n-1}}{m^{n-1}} \right) \left(1 - \left(\frac{\Delta}{p} \right) \right) \right) \\ & \equiv \frac{\binom{2n-1}{n-1}}{m^{n-1}} \left(1 - m^{p-1} + \frac{m-2}{2} u_{p-(\frac{\Delta}{p})}(m-2, 1) \right) \pmod{p^2}, \end{aligned} \quad (1.6)$$

hence

$$\begin{aligned} & \frac{1}{n} \left(\sum_{k=0}^{pn-1} \frac{C_k}{m^k} - \left(\frac{\Delta}{p} \right) \sum_{r=0}^{n-1} \frac{C_r}{m^r} + \left(\frac{\binom{2n-1}{n-1}}{m^{n-1}} - \frac{m}{2} \right) \left(1 - \left(\frac{\Delta}{p} \right) \right) \right) \\ & \equiv \frac{\binom{2n-1}{n-1}}{m^{n-1}} \left(m^{p-1} - 1 + \frac{4-m}{2} u_{p-(\frac{\Delta}{p})}(m-2, 1) \right) \pmod{p^2}. \end{aligned} \quad (1.7)$$

Corollary 1.1. *Let p be an odd prime and let $n \in \mathbb{Z}^+$. Then*

$$\frac{1}{n} \left(\sum_{k=0}^{pn-1} \binom{2k}{k} - \left(\frac{p}{3} \right) \sum_{r=0}^{n-1} \binom{2r}{r} \right) \equiv 0 \pmod{p^2}, \quad (1.8)$$

$$\frac{1}{n} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{2^k} - \left(\frac{-1}{p} \right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{2^r} \right) \equiv 0 \pmod{p^2}, \quad (1.9)$$

and

$$\frac{1}{n} \left(\sum_{k=0}^{pn-1} C_k - \left(\frac{p}{3} \right) \sum_{r=0}^{n-1} C_r + \frac{1-(\frac{p}{3})}{2} \left(\binom{2n}{n} - 1 \right) \right) \equiv 0 \pmod{p^2}. \quad (1.10)$$

When $p > 3$, we also have

$$\frac{1}{n} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{3^k} - \left(\frac{p}{3} \right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{3^r} \right) \equiv 0 \pmod{p^2}. \quad (1.11)$$

Proof. By induction, $u_{2k}(0, 1) = 0$ and $u_{3k}(\pm 1, 1) = 0$ for all $k \in \mathbb{N}$. Applying (1.5) with $m = 1, 2, 3$ we obtain (1.8), (1.9) and (1.11). (1.7) with $m = 1$ yields (1.10). \square

Remark 1.1. Our (1.11) implies (1.3) since $\sum_{r=0}^{n-1} (3r+1)C_r = \binom{2n}{n} - 1$ for all $n \in \mathbb{Z}^+$. (1.9) and (1.10) in the case $n = 1$ were first proved by the author [S11b].

For given integers A and B , the sequence $v_n = v_n(A, B)$ ($n = 0, 1, 2, \dots$) defined by

$$v_0 = 2, \quad v_1 = A, \quad \text{and} \quad v_{n+1} = Av_n - Bv_{n-1} \quad (n = 1, 2, 3, \dots)$$

is called the companion sequence of the Lucas sequence $u_n = u_n(A, B)$ ($n \in \mathbb{N}$). By induction,

$$v_n(A, B) = 2u_{n+1}(A, B) - Au_n(A, B) \quad \text{for all } n \in \mathbb{N}.$$

To prove Theorem 1.1, we need the following auxiliary result on general Lucas sequences which has its own interest.

Theorem 1.2. *Let $A, B \in \mathbb{Z}$ and $\Delta = A^2 - 4B$. Let p be an odd prime and let $n \in \mathbb{Z}^+$. Then*

$$\frac{u_{pn}(A, B) - \left(\frac{\Delta}{p}\right)u_n(A, B)}{pn} \in \mathbb{Z}_p \quad \text{and} \quad \frac{v_{pn}(A, B) - v_n(A, B)}{pn} \in \mathbb{Z}_p, \quad (1.12)$$

where \mathbb{Z}_p denotes the ring of p -adic integers. Moreover, if $p \nmid B\Delta$ then

$$\begin{aligned} \frac{u_{pn}(A, B) - \left(\frac{\Delta}{p}\right)u_n(A, B)}{pn} &\equiv \frac{u_n(A, B)}{2} \left(\frac{\Delta}{p}\right) \frac{B^{p-1} - 1}{p} \\ &\quad + \frac{v_n(A, B)}{2B^{(1 - (\frac{\Delta}{p}))/2}} \cdot \frac{u_{p - (\frac{\Delta}{p})}(A, B)}{p} \pmod{p} \end{aligned} \quad (1.13)$$

and

$$\begin{aligned} \frac{v_{pn}(A, B) - v_n(A, B)}{pn} &\equiv \frac{v_n(A, B)}{2} \cdot \frac{B^{p-1} - 1}{p} \\ &\quad + \frac{\Delta u_n(A, B)}{2B^{(1 - (\frac{\Delta}{p}))/2}} \left(\frac{\Delta}{p}\right) \frac{u_{p - (\frac{\Delta}{p})}(A, B)}{p} \pmod{p}. \end{aligned} \quad (1.14)$$

Remark 1.2. (1.12) in the case $n = 1$ is well known, see, e.g., [S10, Lemma 2.3]. For the prime $p = 2$, (1.12) also holds if we adopt the Kronecker symbol

$$\left(\frac{\Delta}{2}\right) = \begin{cases} 1 & \text{if } \Delta \equiv 1 \pmod{8}, \\ -1 & \text{if } \Delta \equiv 5 \pmod{8}, \\ 0 & \text{if } \Delta \equiv 0 \pmod{2}. \end{cases}$$

Motivated by Theorems 1.1 and 1.2, we give another theorem on supercongruences.

Theorem 1.3. (i) *For any prime $p > 3$ and $n \in \mathbb{Z}^+$, we have*

$$\frac{T_{pn} - T_n}{n} \equiv 0 \pmod{p^2}, \quad (1.15)$$

and moreover

$$T_p \equiv 1 + \frac{p^2}{6} \left(\frac{p}{3}\right) B_{p-2} \left(\frac{1}{3}\right) \pmod{p^3}, \quad (1.16)$$

where the central trinomial coefficient T_n is the coefficient of x^n in the expansion of $(x^2 + x + 1)^n$, and $B_{p-2}(x)$ is the Bernoulli polynomial of degree $p - 2$.

(ii) *For any prime $p > 5$ and $n \in \mathbb{Z}^+$, we have*

$$\frac{g_{pn}(-1) - g_n(-1)}{n^2} \equiv 0 \pmod{p^3}, \quad (1.17)$$

where

$$g_n(x) := \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k} x^k. \quad (1.18)$$

Remark 1.3. (i) The central trinomial coefficients play important roles in combinatorics (for example, T_n is the number of lattice paths from the point $(0, 0)$ to $(n, 0)$ with only allowed steps $(1, 1)$, $(1, -1)$ and $(1, 0)$), see [S14] for some congruences for sums involving generalized central trinomial coefficients. The central trinomial coefficients can be computed in an efficient way since

$$T_0 = T_1 = 1, \quad \text{and } nT_n = (2n-1)T_{n-1} + 3(n-1)T_{n-2} \quad \text{for all } n = 2, 3, \dots$$

By (1.16), $T_p \equiv 1 \pmod{p^2}$ for any prime $p > 3$. We conjecture that $T_n \equiv 1 \pmod{n^2}$ for no composite number $n > 1$, and verify this for n up to 8×10^5 . This conjecture, if true, provides an interesting characterization of primes via central trinomial coefficients. In view of (1.16), for a prime $p > 3$, we have $T_p \equiv 1 \pmod{p^3}$ if and only if $B_{p-2}(1/3) \equiv 0 \pmod{p}$. The author [S15] found that 205129 is the only prime $p < 2 \times 10^7$ for which $B_{p-2}(1/3) \equiv 0 \pmod{p}$.

(ii) The polynomials $g_n(x)$ ($n = 0, 1, 2, \dots$) were introduced by the author [S16] in which the author proved for any prime $p > 5$ that

$$\sum_{k=1}^{p-1} \frac{g_k(-1)}{k} \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{g_k(-1)}{k^2} \equiv 0 \pmod{p};$$

see also V.J. Guo, G.-S. Mao and H. Pan [GMP] for some congruences involving the polynomials $g_n(x)$ ($n = 0, 1, 2, \dots$).

We will show Theorems 1.2, 1.1 and 1.3 in Sections 2, 3 and 4 respectively. In Section 5 we pose some conjectures for further research.

2. PROOF OF THEOREM 1.2

Let $A, B \in \mathbb{Z}$, and let

$$\alpha = \frac{A + \sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A - \sqrt{\Delta}}{2} \quad (2.1)$$

be the two roots of the quadratic equation $x^2 - Ax + B = 0$, where $\Delta = A^2 - 4B$. It is well known that

$$(\alpha - \beta)u_n(A, B) = \alpha^n - \beta^n \quad \text{and} \quad v_n(A, B) = \alpha^n + \beta^n \quad \text{for all } n \in \mathbb{N}. \quad (2.2)$$

When $\Delta = 0$, by induction we have $u_n(A, B) = n(A/2)^{n-1}$ for all $n \in \mathbb{Z}^+$.

Lemma 2.1. *Let $A, B \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Then*

$$u_n(A, B) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-k}{k} A^{n-1-2k} (-B)^k \quad (2.3)$$

and

$$v_n(A, B) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} A^{n-2k} (-B)^k. \quad (2.4)$$

Remark 2.1. Lemma 2.1 is a well known result (see, e.g., [G, (1.60)]) and the two identities (2.3) and (2.4) can be easily proved by induction on n .

Lemma 3.2 [S11a, Lemma 2.2]. *Let $A, B \in \mathbb{Z}$ and let $d \in \mathbb{Z}^+$ be an odd divisor of $\Delta = A^2 - 4B$. Then, for any $n \in \mathbb{Z}^+$, we have*

$$\frac{u_n(A, B)}{n} \equiv \left(\frac{A}{2}\right)^{n-1} + \begin{cases} (A/2)^{n-3} \Delta/3 \pmod{d} & \text{if } 3 \mid d \text{ and } 3 \mid n, \\ 0 \pmod{d} & \text{otherwise.} \end{cases} \quad (2.5)$$

Lemma 2.3 [S12c, Lemma 2.2]. *Let $A, B \in \mathbb{Z}$ and $\Delta = A^2 - 4B$. Suppose that p is an odd prime with $p \nmid B\Delta$. Then we have the congruence*

$$\left(\frac{A \pm \sqrt{\Delta}}{2}\right)^{p - (\frac{\Delta}{p})} \equiv B^{(1 - (\frac{\Delta}{p}))/2} \pmod{p} \quad (2.6)$$

in the ring of algebraic integers.

Lemma 2.4. *Let $A, B \in \mathbb{Z}$ and $\Delta = A^2 - 4B$. For any odd prime p not dividing $B\Delta$, we have*

$$v_{p - (\frac{\Delta}{p})}(A, B) \equiv B^{(1 - (\frac{\Delta}{p}))/2} (B^{p-1} + 1) \pmod{p^2}. \quad (2.7)$$

Proof. By the proof of [S13b, Lemma 3.2],

$$v_{p - (\frac{\Delta}{p})}(A, B) \equiv 2 \left(\frac{B}{p}\right) B^{(p - (\frac{\Delta}{p}))/2} \pmod{p^2}.$$

Thus

$$\begin{aligned} & v_{p - (\frac{\Delta}{p})}(A, B) - B^{(1 - (\frac{\Delta}{p}))/2} (B^{p-1} + 1) \\ & \equiv 2 \left(\frac{B}{p}\right) B^{(p - (\frac{\Delta}{p}))/2} - B^{(1 - (\frac{\Delta}{p}))/2} (B^{p-1} + 1) \\ & = B^{(1 - (\frac{\Delta}{p}))/2} \left(2 \left(\left(\frac{B}{p}\right) B^{(p-1)/2} - 1\right) - (B^{p-1} - 1) \right) \\ & \equiv 0 \pmod{p^2} \end{aligned}$$

since

$$\begin{aligned} B^{p-1} - 1 & = \left(\left(\frac{B}{p}\right) B^{(p-1)/2} + 1\right) \left(\left(\frac{B}{p}\right) B^{(p-1)/2} - 1\right) \\ & \equiv 2 \left(\left(\frac{B}{p}\right) B^{(p-1)/2} - 1\right) \pmod{p^2}. \end{aligned}$$

This concludes the proof. \square

Lemma 2.5. *Let p be a prime and let $n \in \mathbb{Z}^+$. For any p -adic integer $a \not\equiv 0 \pmod{p}$ and positive integer n , we have*

$$\frac{a^{(p-1)n} - 1}{pn} \in \mathbb{Z}_p \quad (2.8)$$

and moreover

$$\frac{a^{(p-1)n} - 1}{pn} \equiv n^{\delta_{p,2}} \frac{a^{p-1} - 1}{p} \pmod{p}. \quad (2.9)$$

Proof. Let r be the unique integer in $\{1, \dots, p-1\}$ with $a \equiv r \pmod{p}$. Then $a^{p-1} \equiv r^{p-1} \equiv 1 \pmod{p}$ by Fermat's little theorem. Write $a^{p-1} = 1 + pt$ with $t \in \mathbb{Z}_p$. Observe that

$$\begin{aligned} \frac{a^{(p-1)n} - 1}{pn} &= \frac{(1+pt)^n - 1}{pn} = \frac{1}{pn} \sum_{k=1}^n \binom{n}{k} (pt)^k = \sum_{k=1}^n \binom{n-1}{k-1} \frac{p^{k-1}}{k} t^k \\ &\equiv t + (n-1) \frac{p}{2} t^2 \equiv t + (n-1) \frac{p}{2} t \equiv n^{\delta_{p,2}} t \pmod{p} \end{aligned}$$

since $p^{k-2}/k \in \mathbb{Z}_p$ for all $k = 3, 4, \dots$. This concludes the proof. \square

Proof of Theorem 1.2. For the sake of brevity, we just write $u_k = u_k(A, B)$ and $v_k = v_k(A, B)$ for all $k \in \mathbb{N}$. Let α and β be the algebraic integers defined by (2.1).

If $p \mid \Delta$, then by Lemma 2.2 we have

$$\frac{u_{pn} - (\frac{\Delta}{p})u_n}{pn} = \frac{u_{pn}}{pn} \equiv \left(\frac{A}{2}\right)^{pn-1} + \delta_{p,3} \left(\frac{A}{2}\right)^{3n-3} \frac{\Delta}{3} \pmod{p}.$$

By (2.2),

$$v_n = \left(\frac{A + \sqrt{\Delta}}{2}\right)^n + \left(\frac{A - \sqrt{\Delta}}{2}\right)^n = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} A^{n-2k} \Delta^k.$$

When $p \mid \Delta$, we have

$$\frac{v_n - A^n/2^{n-1}}{pn} = \frac{\Delta}{p} \sum_{0 < k \leq \lfloor n/2 \rfloor} \binom{n-1}{2k-1} A^{n-2k} \frac{\Delta^{k-1}}{k 2^n} \in \mathbb{Z}_p$$

and similarly

$$\frac{v_{pn} - A^{pn}/2^{pn-1}}{pn} \in \mathbb{Z}_p,$$

hence $(v_{pn} - v_n)/(pn) \in \mathbb{Z}_p$ since

$$\frac{A^{pn}/2^{pn-1} - A^n/2^{n-1}}{pn} = \frac{A^n}{2^{n-1}pn} \left(\left(\frac{A}{2}\right)^{(p-1)n} - 1 \right) \in \mathbb{Z}_p$$

by Lemma 2.5.

Below we assume that $p \nmid \Delta$. Note that

$$\begin{aligned} u_{pn} &= \frac{\alpha^{pn} - \beta^{pn}}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\alpha - \beta} \cdot \frac{(\alpha^n)^p - (\beta^n)^p}{\alpha^n - \beta^n} \\ &= u_n u_p(\alpha^n + \beta^n, \alpha^n \beta^n) = u_n u_p(v_n, B^n) \end{aligned}$$

and

$$v_{pn} = (\alpha^n)^p + (\beta^n)^p = v_p(\alpha^n + \beta^n, \alpha^n \beta^n) = v_p(v_n, B^n).$$

By Lemma 2.1,

$$u_p(v_n, B^n) = \sum_{k=0}^{(p-1)/2} \binom{p-1-k}{k} v_n^{p-1-2k} (-B^n)^k$$

and

$$v_p(v_n, B^n) = \sum_{k=0}^{(p-1)/2} \frac{p}{p-k} \binom{p-k}{k} v_n^{p-2k} (-B^n)^k.$$

Now suppose that $p \mid B$. Then $(-B^n)^k/(pn) \in \mathbb{Z}_p$ for all $k \in \mathbb{Z}^+$ since $p^{n-1}/n \in \mathbb{Z}_p$. Note also that $(\frac{\Delta}{p}) = (\frac{A^2}{p}) = 1$. Thus

$$\frac{u_{pn} - (\frac{\Delta}{p})u_n}{pn} - \frac{u_n}{pn}(v_n^{p-1} - 1) \in \mathbb{Z}_p \quad \text{and} \quad \frac{v_{pn} - v_n^p}{pn} \in \mathbb{Z}_p.$$

In view of (2.4),

$$v_n - A^n = -Bn \sum_{0 < k \leq [n/2]} \binom{n-k-1}{k-1} \frac{(-B)^{k-1}}{k} A^{n-2k}.$$

Since $p \mid B$ and $p^{k-1}/k \in \mathbb{Z}_p$ for all $k \in \mathbb{Z}^+$, we see that $v_n = A^n + pnt$ for some $t \in \mathbb{Z}_p$. By Lemma 2.5, $(A^{(p-1)n} - 1)/(pn) \in \mathbb{Z}_p$. Therefore $(v_n^{p-1} - 1)/(pn) \in \mathbb{Z}_p$ and hence $(u_{pn} - (\frac{\Delta}{p})u_n)/(pn) \in \mathbb{Z}_p$. Note also that

$$\frac{v_{pn} - v_n}{pn} = \frac{v_{pn} - v_n^p}{pn} + v_n \frac{v_n^{p-1} - 1}{pn} \in \mathbb{Z}_p.$$

Below we suppose that $p \nmid B\Delta$.

Case 1. $(\frac{\Delta}{p}) = 1$.

In this case, by Lemma 2.3 we have $\alpha^{p-1} \equiv \beta^{p-1} \equiv 1 \pmod{p}$ in the ring of algebraic integers. Similar to Lemma 2.5, we have

$$\frac{\alpha^{(p-1)n} - 1}{pn} \equiv \frac{\alpha^{p-1} - 1}{p} \pmod{p} \quad \text{and} \quad \frac{\beta^{(p-1)n} - 1}{pn} \equiv \frac{\beta^{p-1} - 1}{p} \pmod{p}.$$

Therefore

$$\begin{aligned}
 \frac{u_{pn} - (\frac{\Delta}{p})u_n}{pn} &= \frac{\alpha^{pn} - \beta^{pn} - (\alpha^n - \beta^n)}{pn(\alpha - \beta)} \\
 &= \frac{\alpha - \beta}{\Delta} \left(\alpha^n \frac{\alpha^{(p-1)n} - 1}{pn} - \beta^n \frac{\beta^{(p-1)n} - 1}{pn} \right) \\
 &\equiv \frac{1}{\alpha - \beta} \cdot \frac{\alpha^n(\alpha^{p-1} - 1) - \beta^n(\beta^{p-1} - 1)}{p} \\
 &= \frac{(\alpha^n - \beta^n)(\alpha^{p-1} + \beta^{p-1} - 2) + (\alpha^n + \beta^n)(\alpha^{p-1} - \beta^{p-1})}{2p(\alpha - \beta)} \\
 &= \frac{u_n}{2} \cdot \frac{v_{p-1} - 2}{p} + \frac{v_n}{2} \cdot \frac{u_{p-1}}{p} \pmod{p}
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{v_{pn} - v_n}{pn} &= \frac{\alpha^{pn} + \beta^{pn} - (\alpha^n + \beta^n)}{pn} \\
 &= \alpha^n \frac{\alpha^{(p-1)n} - 1}{pn} + \beta^n \frac{\beta^{(p-1)n} - 1}{pn} \\
 &\equiv \frac{\alpha^n(\alpha^{p-1} - 1) + \beta^n(\beta^{p-1} - 1)}{p} \\
 &= \frac{(\alpha^n + \beta^n)(\alpha^{p-1} + \beta^{p-1} - 2) + (\alpha^n - \beta^n)(\alpha^{p-1} - \beta^{p-1})}{2p} \\
 &= \frac{v_n}{2} \cdot \frac{v_{p-1} - 2}{p} + \frac{\Delta u_n}{2} \cdot \frac{u_{p-1}}{p} \pmod{p}.
 \end{aligned}$$

Case 2. $(\frac{\Delta}{p}) = -1$.

In this case, by Lemma 2.3 we have $\alpha^{p+1} \equiv \beta^{p+1} \equiv B \pmod{p}$ in the ring of algebraic integers. Thus

$$\begin{aligned}
 \frac{\alpha^{(p+1)n} - B^n}{pn} &= \frac{1}{pn} \sum_{k=1}^n \binom{n}{k} (\alpha^{p+1} - B)^k B^{n-k} \\
 &= \frac{\alpha^{p+1} - B}{p} \sum_{k=1}^n \binom{n-1}{k-1} \frac{(\alpha^{p+1} - B)^{k-1}}{k} B^{n-k} \\
 &\equiv \frac{\alpha^{p+1} - B}{p} B^{n-1} \pmod{p}.
 \end{aligned}$$

Similarly,

$$\frac{\beta^{(p+1)n} - B^n}{pn} \equiv \frac{\beta^{p+1} - B}{p} B^{n-1} \pmod{p}.$$

Therefore

$$\begin{aligned}
\frac{u_{pn} - (\frac{\Delta}{p})u_n}{pn} &= \frac{\alpha^{pn} - \beta^{pn} + \alpha^n - \beta^n}{pn(\alpha - \beta)} \\
&= \frac{\alpha - \beta}{pnB^n\Delta} ((\alpha\beta)^n\alpha^{pn} - (\alpha\beta)^n\beta^{pn} + B^n(\alpha^n - \beta^n)) \\
&= \frac{\alpha - \beta}{B^n\Delta} \left(\beta^n \frac{\alpha^{(p+1)n} - B^n}{pn} + \alpha^n \frac{B^n - \beta^{(p-1)n}}{pn} \right) \\
&\equiv \frac{1}{(\alpha - \beta)B} \left(\beta^n \frac{\alpha^{p+1} - B}{p} + \alpha^n \frac{B - \beta^{p+1}}{p} \right) \\
&= \frac{v_n}{2B} \cdot \frac{u_{p+1}}{p} - \frac{u_n}{2B} \cdot \frac{v_{p+1} - 2B}{p} \pmod{p}
\end{aligned}$$

and

$$\begin{aligned}
\frac{v_{pn} - v_n}{pn} &= \frac{(\alpha\beta)^n}{pnB^n} (\alpha^{pn} + \beta^{pn}) - \frac{\alpha^n + \beta^n}{pn} \\
&= \frac{1}{B^n} \left(\beta^n \frac{\alpha^{(p+1)n} - B^n}{pn} + \alpha^n \frac{\beta^{(p+1)n} - B^n}{pn} \right) \\
&\equiv \frac{1}{B} \left(\beta^n \frac{\alpha^{p+1} - B}{p} + \alpha^n \frac{\beta^{p+1} - B}{p} \right) \\
&= \frac{(\alpha^n + \beta^n)(\alpha^{p+1} + \beta^{p+1} - 2B) - (\alpha^n - \beta^n)(\alpha^{p+1} - \beta^{p+1})}{2Bp} \\
&= \frac{v_n}{2B} \cdot \frac{v_{p+1} - 2B}{p} - \frac{\Delta u_n}{2B} \cdot \frac{u_{p+1}}{p} \pmod{p}.
\end{aligned}$$

Whether $(\frac{\Delta}{p})$ is 1 or -1 , we always have

$$\begin{aligned}
\frac{u_{pn} - (\frac{\Delta}{p})u_n}{pn} &\equiv \left(\frac{\Delta}{p} \right) \frac{u_n}{2B^{(1-(\frac{\Delta}{p}))}/2}} \cdot \frac{v_{p-(\frac{\Delta}{p})} - 2B^{(1-(\frac{\Delta}{p}))}/2}}{p} \\
&\quad + \frac{v_n}{2B^{(1-(\frac{\Delta}{p}))}/2}} \cdot \frac{u_{p-(\frac{\Delta}{p})}}{p} \pmod{p}
\end{aligned}$$

and

$$\begin{aligned}
\frac{v_{pn} - v_n}{pn} &\equiv \frac{v_n}{2B^{(1-(\frac{\Delta}{p}))}/2}} \cdot \frac{v_{p-(\frac{\Delta}{p})} - 2B^{(1-(\frac{\Delta}{p}))}/2}}{p} \\
&\quad + \left(\frac{\Delta}{p} \right) \frac{\Delta u_n}{2B^{(1-(\frac{\Delta}{p}))}/2}} \cdot \frac{u_{p-(\frac{\Delta}{p})}}{p} \pmod{p}.
\end{aligned}$$

By Lemma 2.4,

$$\frac{v_{p-(\frac{\Delta}{p})} - 2B^{(1-(\frac{\Delta}{p}))}/2}}{p} \equiv B^{(1-(\frac{\Delta}{p}))}/2} \frac{B^{p-1} - 1}{p} \pmod{p}.$$

So we have the desired (1.13) and (1.14).

In view of the above, we have completed the proof of Theorem 1.2. \square

3. PROOF OF THEOREM 1.1

Lemma 3.1. *Let $A, B \in \mathbb{Z}$. For any $k, l \in \mathbb{N}$ with $k \geq l$, we have*

$$u_k(A, B)v_l(A, B) - u_l(A, B)v_k(A, B) = 2B^l u_{k-l}(A, B) \quad (3.1)$$

and

$$v_k(A, B)v_l(A, B) - \Delta u_k(A, B)u_l(A, B) = 2B^l v_{k-l}(A, B). \quad (3.2)$$

Proof. (i) Clearly (4.1) holds for $l = 0$. If $k \in \mathbb{Z}^+$, then

$$\begin{aligned} & u_k(A, B)v_1(A, B) - u_1(A, B)v_k(A, B) \\ &= Au_k(A, B) - v_k(A, B) = Au_k(A, B) - (2u_{k+1}(A, B) - Au_k(A, B)) \\ &= 2(Au_k(A, B) - u_{k+1}(A, B)) = 2Bu_{k-1}(A, B). \end{aligned}$$

Now let $k \geq l \geq 2$ and assume that for each $j = 1, 2$ the identity (3.1) with l replaced by $l - j$ still holds. Then

$$\begin{aligned} & u_k(A, B)v_l(A, B) - u_l(A, B)v_k(A, B) \\ &= u_k(A, B)(Av_{l-1}(A, B) - Bv_{l-2}(A, B)) - (Au_{l-1}(A, B) - Bu_{l-2}(A, B))v_k(A, B) \\ &= A(u_k(A, B)v_{l-1}(A, B) - u_{l-1}(A, B)v_k(A, B)) \\ &\quad - B(u_k(A, B)v_{l-2}(A, B) - u_{l-2}(A, B)v_k(A, B)) \\ &= A \times 2B^{l-1}u_{k-(l-1)}(A, B) - B \times 2B^{l-2}u_{k-(l-2)}(A, B) \\ &= 2B^{l-1}(Au_{k-l+1}(A, B) - u_{k-l+2}(A, B)) = 2B^l u_{k-l}(A, B). \end{aligned}$$

This proves (3.1) by induction on l .

(ii) We prove (3.2) in another way. Let α and β be the two roots of the equation $x^2 - Ax + B = 0$. Then $\alpha\beta = B$ and $\Delta = (\alpha - \beta)^2$. Hence

$$\begin{aligned} & v_k(A, B)v_l(A, B) - \Delta u_k(A, B)u_l(A, B) \\ &= (\alpha^k + \beta^k)(\alpha^l + \beta^l) - (\alpha^k - \beta^k)(\alpha^l - \beta^l) \\ &= 2(\alpha^k\beta^l + \alpha^l\beta^k) = 2(\alpha\beta)^l(\alpha^{k-l} + \beta^{k-l}) = 2B^l v_{k-l}(A, B). \end{aligned}$$

This concludes the proof. \square

Lemma 3.2. *For any $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, we have*

$$\sum_{r=0}^{n-1} \binom{2n}{r} u_{n-r}(m-2, 1) = \sum_{k=0}^{n-1} \binom{2k}{k} m^{n-1-k}, \quad (3.3)$$

$$\sum_{r=0}^{n-1} \binom{2n}{r} v_{n-r}(m-2, 1) = m^n - \binom{2n}{n}, \quad (3.4)$$

$$\sum_{r=0}^{n-1} \binom{2n-1}{r} u_{n-r}(m-2, 1) = \frac{1}{2} \sum_{k=0}^{n-1} \binom{2k}{k} m^{n-1-k} + \frac{m^{n-1}}{2}, \quad (3.5)$$

$$\sum_{r=0}^{n-1} \binom{2n-1}{r} v_{n-r}(m-2, 1) = \frac{m-4}{2} \sum_{k=0}^{n-1} \binom{2k}{k} m^{n-1-k} + \frac{m^n}{2} \quad (3.6)$$

Proof. [ST11, (2.1)] with $d = 0$ yields (3.3). Also, [ST11, (2.1)] with $d = 1$ gives

$$\sum_{r=0}^n \binom{2n}{r} u_{n+1-r} = \sum_{k=0}^{n-1} \binom{2k}{k+1} m^{n-1-k} + m^n. \quad (3.7)$$

Hence

$$\begin{aligned} \sum_{r=0}^{n-1} \binom{2n}{r} v_{n-r} &= \sum_{r=0}^{n-1} \binom{2n}{r} (2u_{n+1-r} - (m-2)u_{n-r}) \\ &= 2 \left(\sum_{k=0}^{n-1} \binom{2k}{k+1} m^{n-1-k} + m^n - \binom{2n}{n} \right) \\ &\quad - (m-2) \sum_{k=0}^{n-1} \binom{2k}{k} m^{n-1-k} \\ &= 2m^n - 2 \binom{2n}{n} + \sum_{k=0}^{n-1} \left(2 \binom{2k}{k+1} + (2-m) \binom{2k}{k} \right) m^{n-1-k} \end{aligned}$$

and thus (3.4) follows since

$$\begin{aligned} &\sum_{k=0}^{n-1} \left(2 \binom{2k+1}{k+1} - m \binom{2k}{k} \right) m^{n-1-k} \\ &= \sum_{k=0}^{n-1} \binom{2(k+1)}{k+1} m^{n-(k+1)} - \sum_{k=0}^{n-1} \binom{2k}{k} m^{n-k} = \binom{2n}{n} - m^n. \end{aligned}$$

Substituting $n-1$ for n in (3.3) and (3.7), we get

$$\sum_{r=0}^{n-1} \binom{2(n-1)}{r} u_{n-1-r}(m-2, 1) = \sum_{0 \leq k < n-1} \binom{2k}{k} m^{n-2-k}$$

and

$$\sum_{s=0}^{n-1} \binom{2(n-1)}{s} u_{(n-1)+1-s}(m-2, 1) = \sum_{0 \leq k < n-1} \binom{2k}{k+1} m^{n-2-k} + m^{n-1}.$$

Adding the last two identities and noting that

$$\binom{2k}{k} + \binom{2k}{k+1} = \binom{2k+1}{k} = \frac{1}{2} \binom{2(k+1)}{k+1} \quad \text{for all } k \in \mathbb{N},$$

we obtain

$$\begin{aligned}
 & \sum_{0 \leq k < n-1} \frac{1}{2} \binom{2(k+1)}{k+1} m^{n-2-k} + m^{n-1} \\
 &= \sum_{r=0}^{n-1} \binom{2(n-1)}{r} u_{n-1-r}(m-2, 1) \\
 & \quad + \sum_{0 \leq r < n-1} \binom{2(n-1)}{r+1} u_{n-1-r}(m-2, 1) + u_n(m-2, 1) \\
 &= \sum_{0 \leq r < n-1} \binom{2n-1}{r+1} u_{n-1-r}(m-2, 1) + u_n(m-2, 1)
 \end{aligned}$$

and hence (3.5) holds. (3.3) minus (3.5) gives

$$\sum_{0 < r < n} \binom{2n-1}{r-1} u_{n-r}(m-2, 1) = \frac{1}{2} \sum_{k=0}^{n-1} \binom{2k}{k} m^{n-1-k} - \frac{m^{n-1}}{2}. \quad (3.8)$$

By induction,

$$v_k(m-2, 1) = (m-2)u_k(m-2, 1) - 2u_{k-1}(m-2, 1) \quad \text{for all } k \in \mathbb{Z}^+.$$

Therefore

$$\begin{aligned}
 & \sum_{r=0}^{n-1} \binom{2n-1}{r} v_{n-r}(m-2, 1) \\
 &= (m-2) \sum_{r=0}^{n-1} \binom{2n-1}{r} u_{n-r}(m-2, 1) - 2 \sum_{r=0}^{n-1} \binom{2n-1}{(r+1)-1} u_{n-(r+1)}(m-2, 1) \\
 &= \frac{m-2}{2} \sum_{k=0}^{n-1} \binom{2k}{k} m^{n-1-k} + \frac{m-2}{2} m^{n-1} - \sum_{k=0}^{n-1} \binom{2k}{k} m^{n-1-k} + m^{n-1} \\
 &= \frac{m-4}{2} \sum_{k=0}^{n-1} \binom{2k}{k} m^{n-1-k} + \frac{m^n}{2}
 \end{aligned}$$

with the helps of (3.5) and (3.8). This proves (3.6). \square

Lemma 3.3. *Let $m \in \mathbb{Z} \setminus \{0\}$ and $\Delta = m(m-4)$. And let p be an odd prime with $p \nmid \Delta$. Then*

$$\sum_{k=1}^{p-1} \binom{p}{k} u_k(m-2, 1) \equiv \left(\frac{\Delta}{p}\right) \frac{m^{p-1}-1}{2} + \frac{4-m}{4} u_{p-(\frac{\Delta}{p})}(m-2, 1) \pmod{p^2} \quad (3.9)$$

and

$$\sum_{k=1}^{p-1} \binom{p}{k} v_k(m-2, 1) \equiv \frac{m}{2}(m^{p-1}-1) - \frac{\Delta}{4} \left(\frac{\Delta}{p} \right) u_{p-(\frac{\Delta}{p})}(m-2, 1) \pmod{p^2} \pmod{p^2}. \quad (3.10)$$

Proof. Let α and β be the two roots of the equation $x^2 - (m-2)x + 1 = 0$. Then

$$\begin{aligned} & 2 + v_p(m-2, 1) + \sum_{k=1}^{p-1} \binom{p}{k} v_k(m-2, 1) \\ &= \sum_{k=0}^p \binom{p}{k} (\alpha^k + \beta^k) = (1 + \alpha)^p + (1 + \beta)^p = v_p(m, m). \end{aligned}$$

since

$$(1 + \alpha) + (1 + \beta) = 2 + m - 2 = m$$

and

$$(1 + \alpha)(1 + \beta) = 1 + (\alpha + \beta) + \alpha\beta = 1 + (m-2) + 1 = m.$$

Similarly,

$$\begin{aligned} & u_p(m-2, 1) + \sum_{k=1}^{p-1} \binom{p}{k} u_k(m-2, 1) \\ &= \sum_{k=0}^p \binom{p}{k} \frac{\alpha^k - \beta^k}{\alpha - \beta} = \frac{(1 + \alpha)^p - (1 + \beta)^p}{(1 + \alpha) - (1 + \beta)} = u_p(m, m). \end{aligned}$$

In view of [S10, Lemma 2.4],

$$2u_p(m, m) - \left(\frac{\Delta}{p} \right) m^{p-1} \equiv u_p(m-2, 1) + u_{p-(\frac{\Delta}{p})}(m-2, 1) \pmod{p^2}. \quad (3.11)$$

By [S12b, (3.6)] we have

$$u_p(m-2, 1) - \left(\frac{\Delta}{p} \right) \equiv \frac{m-2}{2} u_{p-(\frac{\Delta}{p})}(m-2, 1) \pmod{p^2}. \quad (3.12)$$

Therefore,

$$\begin{aligned} 2 \sum_{k=1}^{p-1} \binom{p}{k} u_k(m-2, 1) &= 2(u_p(m, m) - u_p(m-2, 1)) \\ &\equiv \left(\frac{\Delta}{p} \right) m^{p-1} - u_p(m-2, 1) + u_{p-(\frac{\Delta}{p})}(m-2, 1) \\ &\equiv \left(\frac{\Delta}{p} \right) (m^{p-1} - 1) + \left(\frac{2-m}{2} + 1 \right) u_{p-(\frac{\Delta}{p})}(m-2, 1) \pmod{p^2}. \end{aligned}$$

This proves (3.9).

By the paragraph following [S10, (2.10)],

$$\left(\frac{\Delta}{p}\right) \frac{v_p(m, m)}{m} \equiv \left(\frac{\Delta}{p}\right) m^{p-1} - (u_p(m, m) - u_p(m-2, 1)) \pmod{p^2}. \quad (3.13)$$

So we have

$$\begin{aligned} \left(\frac{\Delta}{p}\right) \frac{v_p(m, m)}{m} &\equiv \left(\frac{\Delta}{p}\right) \left(m^{p-1} - \frac{m^{p-1} - 1}{2}\right) - \frac{4-m}{4} u_{p-(\frac{\Delta}{p})}(m-2, 1) \\ &= \left(\frac{\Delta}{p}\right) \frac{m^{p-1} + 1}{2} + \frac{m-4}{4} u_{p-(\frac{\Delta}{p})}(m-2, 1) \pmod{p^2}. \end{aligned}$$

As

$$\begin{aligned} v_p(m-2, 1) &= 2u_{p+1}(m-2, 1) - (m-2)u_p(m-2, 1) \\ &= (m-2)u_p(m-2, 1) - 2u_{p-1}(m-2, 1), \end{aligned}$$

we have

$$\begin{aligned} \left(\frac{\Delta}{p}\right) v_p(m-2, 1) &= (m-2)u_p(m-2, 1) - 2u_{p-(\frac{\Delta}{p})}(m-2, 1) \\ &\equiv (m-2) \left(\left(\frac{\Delta}{p}\right) + \frac{m-2}{2} u_{p-(\frac{\Delta}{p})}(m-2, 1) \right) - 2u_{p-(\frac{\Delta}{p})}(m-2, 1) \\ &= (m-2) \left(\frac{\Delta}{p} \right) + \frac{\Delta}{2} u_{p-(\frac{\Delta}{p})}(m-2, 1) \pmod{p^2}. \end{aligned}$$

Combining this with (3.13), we finally get

$$\begin{aligned} \sum_{k=1}^{p-1} \binom{p}{k} v_k(m-2, 1) &= v_p(m, m) - v_p(m-2, 1) - 2 \\ &\equiv \frac{m}{2} (m^{p-1} + 1) + \left(\frac{\Delta}{p}\right) \frac{\Delta}{4} u_{p-(\frac{\Delta}{p})}(m-2, 1) \\ &\quad - (m-2) - \left(\frac{\Delta}{p}\right) \frac{\Delta}{2} u_{p-(\frac{\Delta}{p})}(m-2, 1) - 2 \\ &= \frac{m}{2} (m^{p-1} - 1) - \frac{\Delta}{4} \left(\frac{\Delta}{p}\right) u_{p-(\frac{\Delta}{p})}(m-2, 1) \pmod{p^2}. \end{aligned}$$

This proves (3.10).

In view of the above, we have completed the proof. \square

Lemma 3.4. *Let $m \in \mathbb{Z}$, and let p be an odd prime p not dividing $\Delta = m(m-4)$. Then, for any $n \in \mathbb{Z}^+$ we have*

$$\begin{aligned} & \sum_{r=0}^{n-1} \binom{2n}{r} u_{p(n-r)}(m-2, 1) \\ & \equiv \left(\left(\frac{\Delta}{p} \right) + \frac{m-4}{2} n u_{p-\left(\frac{\Delta}{p}\right)}(m-2, 1) \right) \sum_{k=0}^{n-1} \binom{2k}{k} m^{n-1-k} \\ & \quad + n \binom{2n-1}{n-1} u_{p-\left(\frac{\Delta}{p}\right)}(m-2, 1) \pmod{p^{2+\text{ord}_p(n)}}. \end{aligned} \quad (3.14)$$

Proof. For simplicity, we write $u_k = u_k(m-2, 1)$ and $v_k = v_k(m-2, 1)$ for all $k \in \mathbb{N}$. For each $r = 0, \dots, n$, by Theorem 1.2 we have

$$u_{pr} \equiv \left(\frac{\Delta}{p} \right) u_r + \frac{pr}{2} v_r \frac{u_{p-\left(\frac{\Delta}{p}\right)}}{p} \pmod{p^{2+\text{ord}_p(r)}} \quad (3.15)$$

and

$$v_{pr} \equiv v_r + \frac{pr}{2} \Delta u_r \left(\frac{\Delta}{p} \right) \frac{u_{p-\left(\frac{\Delta}{p}\right)}}{p} \pmod{p^{2+\text{ord}_p(r)}}. \quad (3.16)$$

Since $r \binom{2n}{r} = 2n \binom{2n-1}{r-1}$ for all $r \in \mathbb{Z}^+$, by (3.15) and (3.16) we get

$$\begin{aligned} & \sum_{r=0}^{n-1} \binom{2n}{r} u_{pr} \\ & \equiv \sum_{r=0}^{n-1} \binom{2n}{r} \left(\left(\frac{\Delta}{p} \right) u_r + \frac{r}{2} v_r u_{p-\left(\frac{\Delta}{p}\right)} \right) \\ & = \left(\frac{\Delta}{p} \right) \sum_{r=0}^{n-1} \binom{2n}{r} u_r + n u_{p-\left(\frac{\Delta}{p}\right)} \sum_{0 < r < n} \binom{2n-1}{r-1} v_r \pmod{p^{2+\text{ord}_p(n)}} \end{aligned}$$

and

$$\begin{aligned} & \sum_{r=0}^{n-1} \binom{2n}{r} v_{pr} \\ & \equiv \sum_{r=0}^{n-1} \binom{2n}{r} \left(v_r + \frac{r}{2} \Delta u_r \left(\frac{\Delta}{p} \right) u_{p-\left(\frac{\Delta}{p}\right)} \right) \\ & = \sum_{r=0}^{n-1} \binom{2n}{r} v_r + n \Delta \left(\frac{\Delta}{p} \right) u_{p-\left(\frac{\Delta}{p}\right)} \sum_{0 < r < n} \binom{2n-1}{r-1} u_r \pmod{p^{2+\text{ord}_p(n)}}. \end{aligned}$$

Therefore

$$\begin{aligned}
 & \sum_{r=0}^{n-1} \binom{2n}{r} (u_{pn}v_{pr} - v_{pn}u_{pr}) \\
 & \equiv u_{pn} \sum_{r=0}^{n-1} \binom{2n}{r} v_r - v_{pn} \left(\frac{\Delta}{p} \right) \sum_{r=0}^{n-1} \binom{2n}{r} u_r \\
 & \quad + u_{pn} \times n\Delta \left(\frac{\Delta}{p} \right) u_{p-(\frac{\Delta}{p})} \sum_{0 < r < n} \binom{2n-1}{r-1} u_r \\
 & \quad - v_{pn} \times nu_{p-(\frac{\Delta}{p})} \sum_{0 < r < n} \binom{2n-1}{r-1} v_r \pmod{p^{2+\text{ord}_p(n)}}.
 \end{aligned}$$

In view of (3.15) and (3.16) with $r = n$, from the above we have

$$\begin{aligned}
 & \sum_{r=0}^{n-1} \binom{2n}{r} (u_{pn}v_{pr} - v_{pn}u_{pr}) \\
 & \equiv \left(\left(\frac{\Delta}{p} \right) u_n + \frac{n}{2} v_n u_{p-(\frac{\Delta}{p})} \right) \sum_{r=0}^{n-1} \binom{2n}{r} v_r \\
 & \quad - \left(v_n + \frac{n}{2} \Delta u_n \left(\frac{\Delta}{p} \right) u_{p-(\frac{\Delta}{p})} \right) \left(\frac{\Delta}{p} \right) \sum_{r=0}^{n-1} \binom{2n}{r} u_r \\
 & \quad + nu_n \Delta u_{p-(\frac{\Delta}{p})} \sum_{0 < r < n} \binom{2n-1}{r-1} u_r \\
 & \quad - nv_n u_{p-(\frac{\Delta}{p})} \sum_{0 < r < n} \binom{2n-1}{r-1} v_r \pmod{p^{2+\text{ord}_p(n)}}
 \end{aligned}$$

and hence

$$\begin{aligned}
 & \sum_{r=0}^{n-1} \binom{2n}{r} (u_{pn}v_{pr} - v_{pn}u_{pr}) \\
 & \equiv \left(\frac{\Delta}{p} \right) \sum_{r=0}^{n-1} \binom{2n}{r} (u_n v_r - v_n u_r) \\
 & \quad + \frac{n}{2} u_{p-(\frac{\Delta}{p})} \sum_{r=0}^{n-1} \binom{2n}{r} (v_n v_r - \Delta u_n u_r) \\
 & \quad - nu_{p-(\frac{\Delta}{p})} \sum_{0 < r < n} \binom{2n-1}{r-1} (v_n v_r - \Delta u_n u_r) \pmod{p^{2+\text{ord}_p(n)}}.
 \end{aligned}$$

Thus, with the help of Lemma 3.1 we obtain

$$\begin{aligned}
\sum_{r=0}^{n-1} \binom{2n}{r} u_{pn-pr} &\equiv \left(\frac{\Delta}{p}\right) \sum_{r=0}^{n-1} \binom{2n}{r} u_{n-r} + \frac{n}{2} u_{p-(\frac{\Delta}{p})} \sum_{r=0}^{n-1} \binom{2n}{r} v_{n-r} \\
&\quad - n u_{p-(\frac{\Delta}{p})} \sum_{r=0}^{n-1} \left(\binom{2n}{r} - \binom{2n-1}{r} \right) v_{n-r} \\
&\equiv n u_{p-(\frac{\Delta}{p})} \left(\sum_{r=0}^{n-1} \binom{2n-1}{r} v_{n-r} - \frac{1}{2} \sum_{r=0}^{n-1} \binom{2n}{r} v_{n-r} \right) \\
&\quad + \left(\frac{\Delta}{p}\right) \sum_{r=0}^{n-1} \binom{2n}{r} u_{n-r} \pmod{p^{2+\text{ord}_p(n)}}.
\end{aligned}$$

Combining this with Lemma 3.2, we immediately get (3.14). \square

Lemma 3.5. *For any $m \in \mathbb{Z} \setminus \{0\}$ and $n \in \mathbb{Z}^+$, we have*

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k+1}}{m^k} = \frac{\binom{2n-1}{n-1}}{m^{n-1}} - \frac{m}{2} + \frac{m-2}{2} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{m^k}. \quad (3.17)$$

Proof. Observe that

$$\begin{aligned}
\sum_{k=0}^{n-1} \frac{\binom{2k}{k} + \binom{2k}{k+1}}{m^k} &= \sum_{k=0}^{n-1} \frac{\binom{2k+1}{k}}{m^k} = \frac{\binom{2n-1}{n-1}}{m^{n-1}} + \frac{m}{2} \sum_{0 \leq k < n-1} \frac{\binom{2k+2}{k+1}}{m^{k+1}} \\
&= \frac{\binom{2n-1}{n-1}}{m^{n-1}} + \frac{m}{2} \left(\sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{m^r} - 1 \right).
\end{aligned}$$

So (3.17) follows. \square

Proof of Theorem 1.1. We first handle the case $\Delta = m(m-4) \equiv 0 \pmod{p}$. By [S11a, Theorem 1.1],

$$\frac{1}{pn} \sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{m^k} \equiv \frac{\binom{2pn-1}{pn-1}}{4^{pn-1}} + \delta_{p,3} \frac{m-4}{3} \binom{2n/p^{\text{ord}_p(n)}-1}{n/p^{\text{ord}_p(n)}-1} \pmod{p}.$$

By Lucas' theorem.

$$\binom{2pn-1}{pn-1} = \frac{1}{2} \binom{2pn}{pn} \equiv \frac{1}{2} \binom{2n}{n} \equiv \frac{1}{2} \binom{2n/p^{\text{ord}_p(n)}}{n/p^{\text{ord}_p(n)}} = \binom{2n/p^{\text{ord}_p(n)}-1}{n/p^{\text{ord}_p(n)}-1} \pmod{p}.$$

Since $m \equiv 4 \pmod{p}$, we have $m \equiv 1 \pmod{p}$ if $p = 3$. Therefore

$$\begin{aligned}
\frac{1}{pn} \sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{m^k} &\equiv \frac{\binom{2n-1}{n-1}}{4^{pn-1}} + \delta_{p,3} \frac{m-4}{3} \binom{2n-1}{n-1} \\
&\equiv \frac{\binom{2n-1}{n-1}}{4^{n-1}} + \delta_{p,3} \frac{m(m-4)}{3m^{n-1}} \binom{2n-1}{n-1} \\
&\equiv \frac{\binom{2n-1}{n-1}}{m^{n-1}} \left(1 + \delta_{p,3} \frac{\Delta}{3} \right) \pmod{p}.
\end{aligned}$$

By Lemma 2.2,

$$\frac{u_p(m-2, 1)}{p} \equiv \left(\frac{m-2}{2}\right)^{p-1} + \delta_{p,3} \frac{\Delta}{3} \equiv 1 + \delta_{p,3} \frac{\Delta}{3} \pmod{p}.$$

So

$$\frac{1}{n} \sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{m^k} \equiv \frac{\binom{2n-1}{n-1}}{m^{n-1}} u_p(m-2, 1) \pmod{p^2}$$

as desired.

Below we assume that $p \nmid \Delta$. By [ST11, Lemma 2.1], for any $r = 1, \dots, n$ we have

$$\binom{2pn}{pr} / \binom{2n}{r} \equiv 1 \pmod{p^{2+\text{ord}_p(r)}}$$

and hence

$$\binom{2pn}{pr} \equiv \binom{2n}{r} = \frac{2n}{r} \binom{2n-1}{r-1} \pmod{p^{2+\text{ord}_p(n)}}. \quad (3.18)$$

By Lucas' theorem, for any $r \in \mathbb{N}$ and $k \in \{1, \dots, p\}$ we have

$$\binom{p(2n-1) + p - 1}{pr + k - 1} \equiv \binom{2n-1}{r} \binom{p-1}{k-1} \pmod{p}.$$

So, in view of Lemmas 3.1, 3.2 and 3.4, we have

$$\begin{aligned} & \sum_{k=0}^{pn-1} \binom{2k}{k} m^{pn-1-k} \\ &= \sum_{s=0}^{pn-1} \binom{2pn}{s} u_{pn-s} = \sum_{r=0}^{n-1} \sum_{k=0}^{p-1} \binom{2pn}{pr+k} u_{pn-(pr+k)} \\ &= \sum_{r=0}^{n-1} \binom{2pn}{pr} u_{p(n-r)} + \sum_{r=0}^{n-1} \sum_{k=1}^{p-1} \frac{2pn}{pr+k} \binom{p(2n-1) + p - 1}{pr+k-1} u_{p(n-r)-k} \\ &\equiv \sum_{r=0}^{n-1} \binom{2n}{r} u_{p(n-r)} + \sum_{r=0}^{n-1} \sum_{k=1}^{p-1} \frac{2pn}{k} \binom{2n-1}{r} \binom{p-1}{k-1} u_{p(n-r)-k} \\ &\equiv \left(\left(\frac{\Delta}{p} \right) + \frac{m-4}{2} n u_{p-(\frac{\Delta}{p})} \right) \sum_{k=0}^{n-1} \binom{2k}{k} m^{n-1-k} + n \binom{2n-1}{n-1} u_{p-(\frac{\Delta}{p})} \\ &\quad + n \sum_{r=0}^{n-1} \binom{2n-1}{r} \sum_{k=1}^{p-1} \binom{p}{k} (u_{p(n-r)} v_k - u_k v_{p(n-r)}) \pmod{p^{2+\text{ord}_p(n)}}. \end{aligned}$$

and hence

$$\begin{aligned} \sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{m^k} &\equiv \left(\left(\frac{\Delta}{p} \right) m^{(1-p)n} + \frac{m-4}{2} n u_{p-(\frac{\Delta}{p})} \right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{m^r} + \frac{n}{m^{n-1}} \binom{2n-1}{n-1} u_{p-(\frac{\Delta}{p})} \\ &\quad + \frac{n}{m^{n-1}} \sum_{r=0}^{n-1} \binom{2n-1}{r} \sum_{k=1}^{p-1} \binom{p}{k} (u_{p(n-r)} v_k - u_k v_{p(n-r)}) \pmod{p^{2+\text{ord}_p(n)}}. \end{aligned}$$

By Lemma 2.5,

$$\frac{1}{m^{(p-1)n}} \equiv \frac{1}{1 + n(m^{p-1} - 1)} \equiv 1 - n(m^{p-1} - 1) \pmod{p^{2+\text{ord}_p(n)}}. \quad (3.19)$$

Therefore

$$\begin{aligned} & \frac{1}{n} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{m^k} - \left(\frac{\Delta}{p} \right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{m^r} \right) - \frac{\binom{2n-1}{n-1}}{m^{n-1}} u_{p-(\frac{\Delta}{p})} \\ & \equiv \left(\left(\frac{\Delta}{p} \right) (1 - m^{p-1}) + \frac{m-4}{2} u_{p-(\frac{\Delta}{p})} \right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{m^r} \\ & \quad + \frac{1}{m^{n-1}} \sum_{r=0}^{n-1} \binom{2n-1}{r} \sum_{k=1}^{p-1} \binom{p}{k} (u_{p(n-r)} v_k - u_k v_{p(n-r)}) \pmod{p^2}. \end{aligned} \quad (3.20)$$

Note that $p \mid \binom{p}{k}$ for all $k = 1, \dots, p-1$. In light of Theorem 1.2 and (3.5)-(3.6),

$$\begin{aligned} & \sum_{r=0}^{n-1} \binom{2n-1}{r} \sum_{k=1}^{p-1} \binom{p}{k} (u_{p(n-r)} v_k - u_k v_{p(n-r)}) \\ & \equiv \sum_{r=0}^{n-1} \binom{2n-1}{r} \sum_{k=1}^{p-1} \binom{p}{k} \left(\left(\frac{\Delta}{p} \right) u_{n-r} v_k - u_k v_{n-r} \right) \\ & = \left(\frac{\Delta}{p} \right) \sum_{k=1}^{p-1} \binom{p}{k} v_k \sum_{r=0}^{n-1} \binom{2n-1}{r} u_{n-r} - \sum_{k=1}^{p-1} \binom{p}{k} u_k \sum_{r=0}^{n-1} \binom{2n-1}{r} v_{n-r} \\ & = \left(\frac{\Delta}{p} \right) \sum_{k=1}^{p-1} \binom{p}{k} v_k \left(\frac{1}{2} \sum_{r=0}^{n-1} \binom{2r}{r} m^{n-1-r} + \frac{m^{n-1}}{2} \right) \\ & \quad - \sum_{k=1}^{p-1} \binom{p}{k} u_k \left(\frac{m-4}{2} \sum_{r=0}^{n-1} \binom{2r}{r} m^{n-1-r} + \frac{m^n}{2} \right) \pmod{p^2}. \end{aligned}$$

Combining this with (3.20) we have reduced (1.5) to the congruence

$$\begin{aligned} & \left(\left(\frac{\Delta}{p} \right) (m^{p-1} - 1) + \frac{4-m}{2} u_{p-(\frac{\Delta}{p})} \right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{m^r} \\ & \equiv \left(\frac{\Delta}{p} \right) \sum_{k=1}^{p-1} \binom{p}{k} \frac{v_k}{2} \left(\sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{m^r} + 1 \right) \\ & \quad - \sum_{k=1}^{p-1} \binom{p}{k} \frac{u_k}{2} \left((m-4) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{m^r} + m \right) \pmod{p^2}. \end{aligned}$$

This indeed holds since

$$\left(\frac{\Delta}{p} \right) \sum_{k=1}^{p-1} \binom{p}{k} v_k \equiv m \sum_{k=1}^{p-1} \binom{p}{k} u_k \pmod{p^2}$$

and

$$2 \sum_{k=1}^{p-1} \binom{p}{k} u_k \equiv \left(\frac{\Delta}{p} \right) (m^{p-1} - 1) + \frac{4-m}{2} u_{p-(\frac{\Delta}{p})} \pmod{p^2}$$

by Lemma 3.3. So we finally obtain (1.5).

Next we deduce (1.6) and (1.7) via (1.5). By Lemma 3.5,

$$\sum_{k=0}^{pn-1} \frac{\binom{2k}{k+1}}{m^k} = \frac{\binom{2pn-1}{pn-1}}{m^{pn-1}} - \frac{m}{2} + \frac{m-2}{2} \sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{m^k}. \quad (3.21)$$

Since

$$\binom{2pn-1}{pn-1} = \frac{1}{2} \binom{2pn}{pn} \equiv \frac{1}{2} \binom{2n}{n} = \binom{2n-1}{n-1} \pmod{p^{2+\text{ord}_p(n)}}$$

by (3.18), from (3.19), (3.21) and (1.5) we get

$$\begin{aligned} & \sum_{k=0}^{pn-1} \frac{\binom{2k}{k+1}}{m^k} - \frac{\binom{2n-1}{n-1}}{m^{n-1}} (1 - n(m^{p-1} - 1)) + \frac{m}{2} \\ & \equiv \frac{m-2}{2} \left(\frac{\Delta}{p} \right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{m^r} + \frac{m-2}{2} \cdot \frac{n}{m^{n-1}} \binom{2n-1}{n-1} u_{p-(\frac{\Delta}{p})} \pmod{p^{2+\text{ord}_p(n)}}. \end{aligned}$$

Combining this with (3.17) we immediately obtain (1.6). As $C_k = \binom{2k}{k} - \binom{2k}{k+1}$ for $k \in \mathbb{N}$, (1.7) follows from (1.5) and (1.6).

The proof of Theorem 1.1 is now complete. \square

4. PROOF OF THEOREM 1.3

Lemma 4.1. *Let $p > 3$ be a prime. Then, for any integers $n \geq k \geq 0$, we have*

$$\frac{\binom{pn}{pk}}{\binom{n}{k}} \in 1 + p^3 nk(n-k) \mathbb{Z}_p. \quad (4.1)$$

Remark 4.1. This is a useful known result, see, e.g., [RZ].

Lemma 4.2. *Let p be a prime, and let $j \in \mathbb{N}$ and $k \in \{0, \dots, p-1\}$. Then*

$$\binom{2jp+2k}{jp+k} \equiv \binom{2j}{j} \binom{2k}{k} \pmod{p}. \quad (4.2)$$

Proof. If $2k < p$ then (4.2) follows from Lucas' congruence. If $2k \geq p$, then $p \mid \binom{2k}{k}$, and by the Lucas congruence we have

$$\begin{aligned} \binom{2jp+2k}{jp+k} &= \binom{(2j+1)p + (2k-p)}{jp+k} \\ &\equiv \binom{2j+1}{j} \binom{2k-p}{k} = 0 \equiv \binom{2j}{j} \binom{2k}{k} \pmod{p}. \end{aligned}$$

This concludes the proof. \square

Proof of Theorem 1.3. By the multi-nomial theorem, for any $n \in \mathbb{N}$ we obviously have

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k}$$

and hence

$$(-1)^n T_n = \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} (-1)^k$$

by [G, (3.86)].

Let $p > 3$ be a prime and let $n \in \mathbb{Z}^+$ and $d \in \{1, 2\}$. Since

$$\binom{n}{k} nk(n-k) = n^2(n-k) \binom{n-1}{k-1} \quad \text{and} \quad \binom{n}{k}^2 nk(n-k) = n^3 \binom{n-1}{k-1} \binom{n-1}{n-k-1}$$

for any positive integer $k < n$, by Lemma 4.1 we have

$$\binom{pn}{pk}^d - \binom{n}{k}^d \in p^3 n^{d+1} \mathbb{Z}_p \quad \text{for all } k = 0, \dots, n.$$

Thus

$$\sum_{k=0}^n \binom{pn}{pk}^d \binom{2pk}{pk} (-1)^{pk} \equiv \sum_{k=0}^n \binom{n}{k}^d \binom{2pk}{pk} (-1)^k \pmod{p^{3+(d+1)\text{ord}_p(n)}}.$$

For each $k = 1, 2, 3, \dots$, clearly

$$\binom{2pk}{pk} - \binom{2k}{k} \in p^3 k^3 \mathbb{Z}_p$$

by Lemma 4.1, and

$$\binom{n}{k}^d k^d = n^d \binom{n-1}{k-1}^d \equiv 0 \pmod{n^d}.$$

Therefore,

$$\begin{aligned} \sum_{\substack{k=0 \\ p|k}}^n \binom{pn}{k}^d \binom{2k}{k} (-1)^k &= \sum_{k=0}^n \binom{pn}{pk}^d \binom{2pk}{pk} (-1)^{pk} \\ &\equiv \sum_{k=0}^n \binom{n}{k}^d \binom{2k}{k} (-1)^k \pmod{p^{3+d\text{ord}_p(n)}}. \end{aligned}$$

In view of Lemma 4.2,

$$\begin{aligned}
 & \sum_{\substack{k=0 \\ p \nmid k}}^n \binom{pn}{k}^d \binom{2k}{k} (-1)^k \\
 &= \sum_{r=0}^{n-1} \sum_{k=1}^{p-1} \frac{(pn)^d}{(rp+k)^d} \binom{(n-1)p+p-1}{rp+k-1}^d \binom{2rp+2k}{rp+k} (-1)^{rp+k} \\
 &\equiv \sum_{r=0}^{n-1} \sum_{k=1}^{p-1} \frac{(pn)^d}{k^d} \binom{n-1}{r}^d \binom{p-1}{k-1}^d \binom{2r}{r} \binom{2k}{k} (-1)^{r+k} \\
 &= (pn)^d \sum_{r=0}^{n-1} \binom{n-1}{r}^d \binom{2r}{r} (-1)^r \sum_{k=1}^{p-1} \frac{(-1)^{(k-1)d+k}}{k^d} \binom{2k}{k} \pmod{p^{d+1+d \operatorname{ord}_p(n)}}.
 \end{aligned}$$

Note that

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} \equiv 0 \pmod{p}$$

by [PS] or [ST10]. If $p > 5$, then

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} \equiv 0 \pmod{p}$$

by [T].

Combining the above arguments with $d = 1$, we get

$$\begin{aligned}
 (-1)^{pn} T_{pn} &= \sum_{k=0}^{pn} \binom{pn}{k} \binom{2k}{k} (-1)^k \\
 &\equiv \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} (-1)^k = (-1)^n T_n \pmod{p^{2+\operatorname{ord}_p(n)}}
 \end{aligned}$$

and hence (1.15) is valid. The above arguments with $d = 2$ yield that (1.17) holds if $p > 3$.

Now it remains to prove (1.16) for any fixed prime $p > 3$. Let $H_k := \sum_{0 < j \leq k} 1/j$ for all $k \in \mathbb{N}$. Observe that

$$\begin{aligned}
 (-1)^p T_p &= \sum_{k=0}^p \binom{p}{k} \binom{2k}{k} (-1)^k = 1 - \binom{2p}{p} + \sum_{k=1}^{p-1} \frac{p}{k} \left(\prod_{0 < j < k} \frac{p-j}{j} \right) \binom{2k}{k} (-1)^k \\
 &= 1 - \binom{2p}{p} - \sum_{k=1}^{p-1} \frac{p}{k} \left(\prod_{0 < j < k} \left(1 - \frac{p}{j} \right) \right) \binom{2k}{k} \\
 &\equiv 1 - \binom{2}{1} - \sum_{k=1}^{p-1} \frac{p}{k} (1 - pH_{k-1}) \binom{2k}{k} \\
 &= -1 - p \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} + p^2 \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} H_k - p^2 \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2} \pmod{p^3}
 \end{aligned}$$

with the help of Lemma 4.1. By [ST10], [MT] and [MS], we have

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} \equiv 0 \pmod{p^2}, \quad \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2} \equiv \frac{1}{2} \left(\frac{p}{3}\right) B_{p-2} \left(\frac{1}{3}\right) \pmod{p}$$

and

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} H_k \equiv \frac{1}{3} \left(\frac{p}{3}\right) B_{p-2} \left(\frac{1}{3}\right) \pmod{p}$$

respectively. Therefore (1.16) follows.

The proof of Theorem 1.3 is now complete. \square

5. SOME CONJECTURES

Conjecture 5.1. *Let p be an odd prime. For any integer $m \not\equiv 0 \pmod{p}$ and positive integer n , we have*

$$\frac{1}{n \binom{2n-1}{n-1}} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{m^k} - \left(\frac{\Delta}{p}\right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{m^r} \right) \equiv \frac{u_{p-(\frac{\Delta}{p})}(m-2, 1)}{m^{n-1}} \pmod{p^2}, \quad (5.1)$$

where $\Delta = m(m-4)$.

Remark 5.1. (5.1) is stronger than (1.5).

Conjecture 5.2. (i) *Let p be an odd prime. For any $n \in \mathbb{Z}^+$ we have*

$$\frac{\sum_{k=0}^{pn-1} \binom{2k}{k} - \left(\frac{p}{3}\right) \sum_{r=0}^{n-1} \binom{2r}{r}}{n^2 \binom{2n-1}{n-1}} \equiv \sum_{k=0}^{p-1} \binom{2k}{k} - \left(\frac{p}{3}\right) \pmod{p^4}. \quad (4.2)$$

(ii) *Let $p > 3$ be a prime, and let $m \in \{2, 3\}$ and $\Delta = m(m-4)$. Then there is a p -adic integer $c_p^{(m)}$ only depending on p and m such that for any $n \in \mathbb{Z}^+$ we have*

$$\frac{m^{n-1}}{n^2 \binom{2n-1}{n-1}} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{m^k} - \left(\frac{\Delta}{p}\right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}}{m^r} \right) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{m^k} - \left(\frac{\Delta}{p}\right) + p^3 c_p^{(m)}(n-1) \pmod{p^4}. \quad (5.3)$$

Remark 5.2. In 1992 N. Strauss, J. Shallit and D. Zagier [SSZ] proved that

$$\frac{\sum_{k=0}^{n-1} \binom{2k}{k}}{n^2 \binom{2n}{n}} \equiv -1 \pmod{3} \quad \text{for any } n \in \mathbb{Z}^+.$$

In 2011 the author [S11b] showed that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p} \quad \text{for any odd prime } p,$$

where E_0, E_1, E_2, \dots are the Euler numbers defined by

$$\frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} \quad (|x| < \pi).$$

Conjecture 5.3. (i) Let p be an odd prime. For any integer $m \not\equiv 0 \pmod{p}$ and positive integer n , we have

$$\frac{1}{pn} \left(\sum_{k=0}^{pn-1} \binom{pn-1}{k} \frac{\binom{2k}{k}}{(-m)^k} - \left(\frac{m(m-4)}{p} \right) \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\binom{2r}{r}}{(-m)^r} \right) \in \mathbb{Z}_p. \quad (5.4)$$

(ii) For any prime $p \neq 3$ and $n \in \mathbb{Z}^+$, we have

$$\frac{1}{p^2 n^2} \left(\sum_{k=0}^{pn-1} \binom{pn-1}{k} \frac{\binom{2k}{k}}{(-3)^k} - \left(\frac{p}{3} \right) \sum_{r=0}^{n-1} \binom{n-1}{r} \frac{\binom{2r}{r}}{(-3)^r} \right) \in \mathbb{Z}_p. \quad (5.5)$$

Remark 5.3. The author [S12b] determined $\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k} / (-m)^k$ modulo p^2 for any odd prime p and integer $m \not\equiv 0 \pmod{p}$.

Conjecture 5.4. Let $p > 3$ be a prime and let $n \in \mathbb{Z}^+$. Then

$$\frac{16^n}{n^2 \binom{2n}{n}^2} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k}}{16^k} - \left(\frac{-1}{p} \right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}^2}{16^r} \right) \equiv -4p^2 E_{p-3} \pmod{p^3}, \quad (5.6)$$

$$\frac{27^n}{n^2 \binom{2n}{n} \binom{3n}{n}} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} - \left(\frac{p}{3} \right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r} \binom{3r}{r}}{27^r} \right) \equiv -\frac{3}{2} p^2 B_{p-2} \left(\frac{1}{3} \right) \pmod{p^3}, \quad (5.7)$$

$$\frac{64^n}{n^2 \binom{4n}{2n} \binom{2n}{n}} \left(\sum_{k=0}^{pn-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} - \left(\frac{-2}{p} \right) \sum_{r=0}^{n-1} \frac{\binom{4r}{2r} \binom{2r}{r}}{64^r} \right) \equiv -p^2 E_{p-3} \left(\frac{1}{4} \right) \pmod{p^3}, \quad (5.8)$$

$$\frac{432^n}{n^2 \binom{6n}{3n} \binom{3n}{n}} \left(\sum_{k=0}^{pn-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} - \left(\frac{-1}{p} \right) \sum_{r=0}^{n-1} \frac{\binom{6r}{3r} \binom{3r}{r}}{432^r} \right) \equiv -20p^2 E_{p-3} \pmod{p^3}, \quad (5.9)$$

where $E_{p-3}(x)$ is the Euler polynomial of degree $p-3$.

Remark 5.4. Let $p > 3$ be a prime. Recently, J.-C. Liu [L] proved that for any $n \in \mathbb{Z}^+$ we have

$$\begin{aligned} \sum_{k=0}^{pn-1} \frac{\binom{2k}{k}^2}{16^k} &\equiv \left(\frac{-1}{p} \right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r}^2}{16^r} \pmod{p^2}, \\ \sum_{k=0}^{pn-1} \frac{\binom{2k}{k} \binom{3k}{k}}{27^k} &\equiv \left(\frac{p}{3} \right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r} \binom{3r}{r}}{27^r} \pmod{p^2}, \\ \sum_{k=0}^{pn-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{64^k} &\equiv \left(\frac{-2}{p} \right) \sum_{r=0}^{n-1} \frac{\binom{4r}{2r} \binom{2r}{r}}{64^r} \pmod{p^2}, \\ \sum_{k=0}^{pn-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{432^k} &\equiv \left(\frac{-1}{p} \right) \sum_{r=0}^{n-1} \frac{\binom{6r}{3r} \binom{3r}{r}}{432^r} \pmod{p^2}, \end{aligned}$$

the case $n = 1$ of which is a conjecture of F. Rodriguez-Villegas [RV] first confirmed by E. Mortenson [M03]. In the case $n = 1$, (5.6) was established by the author [S11b], and (5.7)-(5.9) were conjectured by the author and later confirmed by Z.-H. Sun [Su2].

Conjecture 5.5. *Let $p > 3$ be a prime and let $n \in \mathbb{Z}^+$. Then*

$$\begin{aligned} & \frac{27^n}{(pn)^2 \binom{2n}{n} \binom{3n}{n}} \left(\sum_{k=0}^{pn-1} \frac{\binom{2k}{k} \binom{3k}{k}}{(2k+1)27^k} - \left(\frac{p}{3}\right) \sum_{r=0}^{n-1} \frac{\binom{2r}{r} \binom{3r}{r}}{(2r+1)27^r} \right) \\ & \equiv -3B_{p-2} \left(\frac{1}{3}\right) \pmod{p} \end{aligned} \quad (5.10)$$

and

$$\frac{27^n}{(pn)^4 \binom{2n}{n} \binom{3n}{n}} \left(\sum_{k=0}^{pn-1} (4k+1) \frac{\binom{2k}{k} \binom{3k}{k}}{(2k+1)27^k} - \left(\frac{p}{3}\right) \sum_{r=0}^{n-1} (4r+1) \frac{\binom{2r}{r} \binom{3r}{r}}{(2r+1)27^r} \right) \in \mathbb{Z}_p. \quad (5.11)$$

Also,

$$\frac{64^n}{(pn)^2 \binom{4n}{2n} \binom{2n}{n}} \left(\sum_{k=0}^{pn-1} \frac{\binom{4k}{2k} \binom{2k}{k}}{(2k+1)64^k} - \left(\frac{-1}{p}\right) \sum_{r=0}^{n-1} \frac{\binom{4r}{2r} \binom{2r}{r}}{(2r+1)64^r} \right) \equiv -16E_{p-3} \pmod{p} \quad (5.12)$$

and

$$\begin{aligned} & \frac{432^n}{(pn)^2 \binom{6n}{3n} \binom{3n}{n}} \left(\sum_{k=0}^{pn-1} \frac{\binom{6k}{3k} \binom{3k}{k}}{(2k+1)432^k} - \left(\frac{p}{3}\right) \sum_{r=0}^{n-1} \frac{\binom{6r}{3r} \binom{3r}{r}}{(2r+1)432^r} \right) \\ & \equiv -\frac{15}{2}B_{p-2} \left(\frac{1}{3}\right) \pmod{p}. \end{aligned} \quad (5.13)$$

Remark 5.5. Those integers

$$C_k^{(2)} := \frac{\binom{3k}{k}}{2k+1} = \binom{3k}{k} - 2 \binom{3k}{k-1} \quad (k = 1, 2, 3, \dots)$$

are called second-order Catalan numbers. In the case $n = 1$, (5.10)-(5.12), as well as the fact that the left-hand side of (5.13) is p -adic integral, were originally conjectured by the author [S11b, Conjecture 5.12]. In 2016 Z.-H. Sun [Su1] confirmed (5.10), (5.12) and (5.13) in the case $n = 1$.

Conjecture 5.6. *For any prime $p > 3$ and $n \in \mathbb{Z}^+$, we have*

$$\frac{48^n}{(pn)^2 \binom{4n}{2n} \binom{2n}{n}} \left(\sum_{k=0}^{pn-1} \frac{\binom{4k}{2k+1} \binom{2k}{k}}{48^k} - \left(\frac{p}{3}\right) \sum_{r=0}^{n-1} \frac{\binom{4r}{2r+1} \binom{2r}{r}}{48^r} \right) \equiv \frac{5}{3}B_{p-2} \left(\frac{1}{3}\right) \pmod{p}. \quad (5.14)$$

Remark 5.6. (5.14) with $n = 1$ was conjectured by the author [S15]. It seems difficult to prove that the left-hand side of (5.14) is p -adic integral.

Conjecture 5.7. *For any odd prime p and positive integer n , we have*

$$\frac{1}{n^3 \binom{2n}{n}^3} \left(\frac{1}{p} \sum_{k=0}^{pn-1} (21k+8) \binom{2k}{k}^3 - \sum_{r=0}^{n-1} (21r+8) \binom{2r}{r}^3 \right) \equiv 0 \pmod{p^3}. \quad (5.15)$$

Remark 5.7. (5.15) in the case $n = 1$ was proved by the author in [S11b]. We guess that all those Ramanujan-type supercongruences should have extensions involving $n \in \mathbb{Z}^+$ similar to (1.15).

It is well known that $\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2$ for all $n \in \mathbb{N}$. The Franel numbers are given by

$$f_n := \sum_{k=0}^n \binom{n}{k}^3 \quad (n = 0, 1, 2, \dots).$$

As in [S16, Conjecture 4.3] we also set

$$F(n) := \sum_{k=0}^n \binom{n}{k}^3 (-8)^k \quad (n = 0, 1, 2, \dots).$$

Conjecture 5.8. *For any odd prime p and positive integer n , we have*

$$\frac{1}{n^2} \left(\sum_{k=0}^{pn-1} (-1)^k f_k - \left(\frac{p}{3} \right) \sum_{r=0}^{n-1} (-1)^r f_r \right) \equiv 0 \pmod{p^2}, \quad (5.16)$$

$$\frac{1}{n^2} \left(\sum_{k=0}^{pn-1} \frac{f_k}{8^k} - \left(\frac{p}{3} \right) \sum_{r=0}^{n-1} \frac{f_r}{8^r} \right) \equiv 0 \pmod{p^2}, \quad (5.17)$$

and

$$\frac{1}{n^2} \left(\sum_{k=0}^{pn-1} (-1)^k F(k) - \left(\frac{p}{3} \right) \sum_{r=0}^{n-1} (-1)^r F(r) \right) \equiv 0 \pmod{p^2}. \quad (5.18)$$

Remark 5.8. In the case $n = 1$, (5.16) was first established by the author [S13a, (1.5)], and (5.17) and (5.18) were conjectured by the author in [S13a, Remark 1.1].

Conjecture 5.9. (i) *Let $p > 3$ be a prime and let $n \in \mathbb{Z}^+$. Then*

$$\frac{T_{pn} - T_n}{(pn)^2} \in \mathbb{Z}_p \quad (5.19)$$

and

$$\frac{T_{p^n} - T_{p^{n-1}}}{p^{2n}} \equiv \left(\frac{p^n}{3} \right) \frac{1}{6} B_{p-2} \left(\frac{1}{3} \right) \pmod{p}. \quad (5.20)$$

(ii) For any prime $p > 5$ and $n \in \mathbb{Z}^+$, we have

$$\frac{g_{pn}(-1) - g_n(-1)}{(pn)^3} \in \mathbb{Z}_p. \quad (5.21)$$

(iii) Let $p > 3$ be a prime and let $n \in \mathbb{Z}^+$. Then

$$\frac{1}{(pn)^3} \left(\sum_{k=0}^{pn-1} G(k) - \sum_{r=0}^{n-1} G(r) \right) \in \mathbb{Z}_p \quad (5.22)$$

and

$$\frac{1}{p^{3n}} \sum_{k=p^{n-1}}^{p^n-1} G(k) \equiv -\frac{4}{3} B_{p-3} \pmod{p},$$

where $G(k) := \sum_{j=0}^k \binom{k}{j}^2 (6j+1) C_j$, and B_0, B_1, B_2, \dots are the Bernoulli numbers.

Remark 5.9. Parts (i) and (ii) are stronger than Theorem 1.3. We note the new identity

$$g_n(-1) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}^2 \binom{2k}{k} (-1)^{n-k}$$

which can be easily proved via the Zeilberger algorithm (cf. [PWZ, pp. 101-119]).

Conjecture 5.10. For $n = 0, 1, 2, \dots$ define

$$\begin{aligned} a_n &:= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} \binom{n-k}{k}, \\ b_n &:= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}^2 \binom{n-k}{k}, \\ c_n &:= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}^2 \binom{n-k}{k}. \end{aligned}$$

Let n be any positive integer. Then

$$\frac{a_{pn} - a_n}{(pn)^2} \in \mathbb{Z}_p \quad \text{for each prime } p > 3. \quad (5.23)$$

Also, for any prime $p > 5$ we have

$$\frac{b_{pn} - b_n}{(pn)^3} \in \mathbb{Z}_p \quad \text{and} \quad \frac{c_{pn} - c_n}{(pn)^3} \in \mathbb{Z}_p. \quad (5.24)$$

Remark 5.10. Let $p > 3$ be a prime and let $n \in \mathbb{Z}^+$. We are able to show that $(a_{pn} - a_n)/(p^2n) \in \mathbb{Z}_p$ which is similar to (1.15). Also, if $p > 5$ then $(b_{pn} - b_n)/(p^2n) \in \mathbb{Z}_p$ and $(c_{pn} - c_n)/(p^2n) \in \mathbb{Z}_p$, which are similar to (1.17).

Recall the definition of $g_n(x)$ given by (1.18). It is known that $g_n := g_n(1)$ coincides with $\sum_{k=0}^n \binom{n}{k} f_k$ (cf. [B]), and an extension of this to polynomials was given by the author [S16, Theorem 2.2]. We also set

$$h_n := \int_0^1 g_n(x) dx = \sum_{k=0}^n \binom{n}{k}^2 C_k \quad (n = 0, 1, 2, \dots).$$

Conjecture 5.11. *Let p be an odd prime p and let $n \in \mathbb{Z}^+$. Then*

$$\frac{1}{(pn)^2} \left(\sum_{k=0}^{pn-1} g_k(-1) - \left(\frac{-1}{p} \right) \sum_{r=0}^{n-1} g_r(-1) \right) \in \mathbb{Z}_p, \quad (5.25)$$

$$\frac{1}{(pn)^2} \left(\sum_{k=0}^{pn-1} g_k(-3) - \left(\frac{p}{3} \right) \sum_{r=0}^{n-1} g_r(-3) \right) \in \mathbb{Z}_p. \quad (5.26)$$

Also,

$$\begin{aligned} \frac{1}{(pn)^2} \left(\sum_{k=0}^{pn-1} g_k - \sum_{r=0}^{n-1} g_r \right) &\equiv \frac{5}{8} \left(\frac{p}{3} \right) B_{p-2} \left(\frac{1}{3} \right) g_{n-1} \\ &+ \begin{cases} -1 \pmod{p} & \text{if } p = 3 \text{ \& } n = 1, \\ 1 \pmod{p} & \text{if } p = n = 3, \\ 0 \pmod{p} & \text{otherwise,} \end{cases} \end{aligned} \quad (5.27)$$

and

$$\begin{aligned} \frac{1}{(pn)^2} \left(\sum_{k=0}^{pn-1} h_k - \sum_{r=0}^{n-1} h_r \right) &\equiv \frac{3}{4} \left(\frac{p}{3} \right) B_{p-2} \left(\frac{1}{3} \right) g_{n-1} \\ &+ \begin{cases} n \pmod{p} & \text{if } p = 3 > n, \\ 0 \pmod{p} & \text{otherwise.} \end{cases} \end{aligned} \quad (5.28)$$

When $p > 3$, we have

$$\frac{1}{(pn)^2} \left(\sum_{k=0}^{pn-1} \frac{g_k}{9^k} - \left(\frac{p}{3} \right) \sum_{r=0}^{n-1} \frac{g_r}{9^r} \right) \equiv -\frac{5}{8} B_{p-2} \left(\frac{1}{3} \right) \frac{g_n}{9^n} \pmod{p}. \quad (5.29)$$

Remark 5.11. (5.25)-(5.27) and (5.28) are extensions of [S16, Theorem 1.1(i)] and [S12a, Corollary 1.5] respectively, and the left-hand side of (5.29) with $n = 1$ was conjectured to be a p -adic integer by the author [S16, Remark 1.1].

Define

$$\bar{P}_n := \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} \quad (n = 0, 1, 2, \dots).$$

Those $P_n = 2^n \bar{P}_n$ with $n \in \mathbb{N}$ are usually called Catalan-Larcombe-French numbers. See [S15] for some congruences and series for $1/\pi$ related to \bar{P}_n .

Conjecture 5.12. *Let p be an odd prime and let $n \in \mathbb{Z}^+$. If $p > 3$ or $3 \nmid n$, then*

$$\frac{1}{(pn)^2} \left(\sum_{k=0}^{pn-1} \frac{\bar{P}_k}{4^k} - \sum_{r=0}^{n-1} \frac{\bar{P}_r}{4^r} \right) \equiv \left(\frac{-1}{p} \right) 2E_{p-3} \frac{\bar{P}_{n-1}}{4^{n-1}} \pmod{p} \quad (5.30)$$

and

$$\frac{1}{(pn)^2} \left(\sum_{k=0}^{pn-1} \frac{\bar{P}_k}{8^k} - \left(\frac{-1}{p} \right) \sum_{r=0}^{n-1} \frac{\bar{P}_r}{8^r} \right) \equiv -2E_{p-3} \frac{\bar{P}_n}{8^n} \pmod{p}. \quad (5.31)$$

Remark 5.12. Conjecture 5.12 in the case $n = 1$ appeared in [S13c, Remark 3.13].

In the same spirit, we have many other conjectures similar to the above ones. In our opinion, almost all previous known congruences should have such extensions involving a parameter $n \in \mathbb{Z}^+$.

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